Defining Algebraic Effects and Handlers (AEH) via Trails and Metacontinuations

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Delimited continuations

For shift:

- clear CPS semantics that everyone can agree with [DF90]
- correctness of CPS translation [DF92]
- type system that corresponds to CPS interpreter [DF89]
- abstract machine derived from the CPS interpreter

They extend to control, shift0, and control0.

We have unified framework for four delimited continuation constructs.

Algebraic effects and handlers (AEH)

Deep and shallow handlers:

- CPS semantics, type systems, and abstract machines exist. But they are developed independently.

(And there are many other variants of AEH.)

How are they related each other?
How are they related to those of delimited continuations?

We should build AEH based on the foundations we already have for delimited continuations.

Plan

Goal

To understand the definitional interpreter for AEH

Definitional Interpreter for AEH

$$\begin{aligned}
\mathcal{E}\,[\![x]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\kappa\,\rho(x)\,t\,m \\
\mathcal{E}\,[\![\lambda x.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\kappa\,(\lambda v.\,\lambda\kappa'.\,\mathcal{E}\,[\![e]\!]\,\rho[v/x]\,\kappa')\,t\,m \\
\mathcal{E}\,[\![e_1\,e_2]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e_1]\!]\,\rho\,(\lambda v_1.\,\lambda t.\,\lambda m. \\
&\qquad \mathcal{E}\,[\![e_2]\!]\,\rho\,(\lambda v_2.\,\lambda t.\,\lambda m.\,v_1\,v_2\,\kappa\,t\,m)\,t\,m)\,tm
\end{aligned}$$

$$\mathcal{E}\,[\![\textsf{try}\;e_1\;\textsf{with}\;(x;k).\,e_2]\!]\,\rho\,\kappa = \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e_1]\!]\,\rho\,id_\kappa\,id_t\,((\kappa, t, h) :: m) \qquad \leftarrow \text{push}$$

$$\text{where}\quad h = \lambda v.\,\lambda v_\kappa.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\mathcal{E}\,[\![e_2]\!]\,[v/x, v_\kappa/k]\,\kappa'\,t'\,m' \qquad \downarrow \text{pop}$$

$$\mathcal{E}\,[\![\textsf{call}(e)]\!]\,\rho\,\kappa = \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e]\!]\,\rho\,(\lambda v.\,\lambda t''.\,\lambda((\kappa_0, t_0, h) :: m_0).\,h\,v\,v_\kappa\,\kappa_0\,t_0\,m_0)\,t\,m$$

$$\text{where}\quad v_\kappa = \lambda v'.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v'\,t''\,((\kappa', t', h) :: m') \qquad \text{(deep)}$$

$$v_\kappa = \lambda v'.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v'\,(t''\,@\,\kappa' :: t')\,m' \qquad \text{(shallow)}$$

Plan

Goal

To understand the definitional interpreter for AEH

via

- reviewing CPS interpreter (for shift)
- adding metacontinuations to remove reset (for shift0), and
- adding trails to remove reset of inv. context (for control/0).

reduction rules:

$$\langle F[\mathcal{S}k.\,e]\rangle \longrightarrow \langle e[\lambda x.\,\langle F[x]\rangle/k]\rangle$$

$$\langle F[\mathcal{S}_0 k.\,e]\rangle \longrightarrow e[\lambda x.\,\langle F[x]\rangle/k]$$

$$\langle F[\mathcal{F}k.\,e]\rangle \longrightarrow \langle e[\lambda x.\,F[x]/k]\rangle$$

$$\langle F[\mathcal{F}_0 k.\,e]\rangle \longrightarrow e[\lambda x.\,F[x]/k]$$

CPS interpreter for shift/reset in CPS

$$\begin{aligned}
\mathcal{E}\,[\![x]\!]\,\rho\,\kappa &= \kappa\,\rho(x) \\
\mathcal{E}\,[\![\lambda x.\,e]\!]\,\rho\,\kappa &= \kappa\,(\lambda v.\,\lambda \kappa'.\,\mathcal{E}\,[\![e]\!]\,\rho[v/x]\,\kappa') \\
\mathcal{E}\,[\![e_1\,e_2]\!]\,\rho\,\kappa &= \mathcal{E}\,[\![e_1]\!]\,\rho\,(\lambda v_1. \\
&\qquad \mathcal{E}\,[\![e_2]\!]\,\rho\,(\lambda v_2.\,v_1\,v_2\,\kappa))
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}\,[\![\mathcal{S}k.\,e]\!]\,\rho\,\kappa &= \mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda \kappa'.\,\kappa'\,(\kappa\,v)/k]\,id_\kappa \\
\mathcal{E}\,[\![\langle e\rangle]\!]\,\rho\,\kappa &= \kappa\,(\mathcal{E}\,[\![e]\!]\,\rho\,id_\kappa)
\end{aligned}$$

$$2 + \langle 3 + \mathcal{S}k.\,(4 \times k\,5)\rangle$$

cf. κ is applied in DS. It cannot be captured. (Also for $\kappa'\,(\kappa\,v)$.)

Metacontinuations to remove reset (for shift0)

$$\mathcal{E}\,[\![x]\!]\,\rho\,\kappa \;=\; \lambda m.\,\kappa\,\rho(x)\,m$$

$$\mathcal{E}\,[\![\lambda x.\,e]\!]\,\rho\,\kappa \;=\; \lambda m.\,\kappa\,(\lambda v.\,\lambda\kappa'.\,\mathcal{E}\,[\![e]\!]\,\rho[v/x]\,\kappa')\,m$$

$$\mathcal{E}\,[\![e_1\,e_2]\!]\,\rho\,\kappa \;=\; \lambda m.\,\mathcal{E}\,[\![e_1]\!]\,\rho\,(\lambda v_1.\,\lambda m.\,\mathcal{E}\,[\![e_2]\!]\,\rho\,(\lambda v_2.\,\lambda m.\,v_1\,v_2\,\kappa\,m)\,m)\,m$$

$$\mathcal{E}\,[\![\mathcal{S}k.\,e]\!]\,\rho\,\kappa \;=\; \lambda m.\;\mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda\kappa'.\,\lambda m.\,\kappa\,v\,(\kappa', m)\,/k]\,id_\kappa\,m$$

$$\mathcal{E}\,[\![\mathcal{S}_0 k.\,e]\!]\,\rho\,\kappa \;=\; \lambda(\kappa_0, m_0).\,\mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda\kappa'.\,\lambda m.\,\kappa\,v\,(\kappa', m)\,/k]\,\kappa_0\,m_0$$

$$\mathcal{E}\,[\![\langle e\rangle]\!]\,\rho\,\kappa \;=\; \lambda m.\,\mathcal{E}\,[\![e]\!]\,\rho\,id_\kappa\,(\kappa, m)$$

$$1 + \langle\, 2 + \langle\, 3 + \mathcal{S}_0 k_1.\,\mathcal{S}_0 k_2.\,(4 \times k_2\,5)\,\rangle\,\rangle$$

cf. κ does not have access to invocation context κ'.

Trails to remove reset of inv. context (for control/0)

$$
\begin{aligned}
\mathcal{E}\,[\![x]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\kappa\,\rho(x)\,t\,m\\
\mathcal{E}\,[\![\lambda x.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\kappa\,(\lambda v.\,\lambda\kappa'.\,\mathcal{E}\,[\![e]\!]\,\rho[v/x]\,\kappa')\,t\,m\\
\mathcal{E}\,[\![e_1\;e_2]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e_1]\!]\,\rho\,(\lambda v_1.\,\lambda t.\,\lambda m.\\
&\qquad \mathcal{E}\,[\![e_2]\!]\,\rho\,(\lambda v_2.\,\lambda t.\,\lambda m.\,v_1\,v_2\,\kappa\,t\,m)\,t\,m)\,t\,m\\
\mathcal{E}\,[\![\mathcal{S}k.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\quad \mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v\,t\,((\kappa',t'),m')\,/k]\,id_\kappa\,id_t\,m\\
\mathcal{E}\,[\![\mathcal{S}_0k.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda((\kappa_0,t_0),m_0).\,\mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v\,t\,((\kappa',t'),m')\,/k]\,\kappa_0\,t_0\,m_0\\
\hline
\mathcal{E}\,[\![\mathcal{F}k.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\quad \mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v\,(t\,@\,\kappa'\,::\,t')\,m'\,/k]\,id_\kappa\,id_t\,m\\
\mathcal{E}\,[\![\mathcal{F}_0k.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda((\kappa_0,t_0),m_0).\,\mathcal{E}\,[\![e]\!]\,\rho[\lambda v.\,\lambda\kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v\,(t\,@\,\kappa'\,::\,t')\,m'\,/k]\,\kappa_0\,t_0\,m_0\\
\mathcal{E}\,[\![\langle e\rangle]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e]\!]\,\rho\,id_\kappa\,id_t\,((\kappa,t),m)
\end{aligned}
$$

$$1 + \langle\, \mathcal{F}_0k_1.\,(2 \times k_1\,3) + \mathcal{F}_0k_2.\,(4 \times k_2\,5)\,\rangle$$

Summary so far

Metacontinuation

a stack of continuations and trails

- $\langle e \rangle$ pushes the current continuation and trail
- $\mathcal{S}_0$ and $\mathcal{F}_0$ pop them to access outside reset

Trail

a list of invocation contexts

- $\langle e \rangle$ installs an empty trail
- $\mathcal{F}$ and $\mathcal{F}_0$ append the invocation context (at the end)

Definitional Interpreter for AEH

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\mathcal{E}\,[\![\lambda x.\,e]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\kappa\,(\lambda v.\,\lambda \kappa'.\,\mathcal{E}\,[\![e]\!]\,\rho[v/x]\,\kappa')\,t\,m\\
\mathcal{E}\,[\![e_1\;e_2]\!]\,\rho\,\kappa &= \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e_1]\!]\,\rho\,(\lambda v_1.\,\lambda t.\,\lambda m.\\
&\qquad \mathcal{E}\,[\![e_2]\!]\,\rho\,(\lambda v_2.\,\lambda t.\,\lambda m.\,v_1\,v_2\,\kappa\,t\,m)\,t\,m)\,tm
\end{aligned}$$

$$\mathcal{E}\,[\![\mathsf{try}\;e_1\;\mathsf{with}\;(x;k).\,e_2]\!]\,\rho\,\kappa = \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e_1]\!]\,\rho\,id_\kappa\,id_t\,((\kappa, t, h) :: m) \qquad \leftarrow \text{push}$$

$$\text{where}\quad h = \lambda v.\,\lambda v_\kappa.\,\lambda \kappa'.\,\lambda t'.\,\lambda m'.\,\mathcal{E}\,[\![e_2]\!]\,[v/x, v_\kappa/k]\,\kappa'\,t'\,m' \qquad \downarrow \text{pop}$$

$$\mathcal{E}\,[\![\mathsf{call}(e)]\!]\,\rho\,\kappa = \lambda t.\,\lambda m.\,\mathcal{E}\,[\![e]\!]\,\rho\,(\lambda v.\,\lambda t''.\,\lambda((\kappa_0, t_0, h) :: m_0).\,h\,v\,v_\kappa\,\kappa_0\,t_0\,m_0)\,t\,m$$

$$\text{where}\quad v_\kappa = \lambda v'.\,\lambda \kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v'\,t''\,((\kappa', t', h) :: m') \qquad \text{(deep)}$$

$$v_\kappa = \lambda v'.\,\lambda \kappa'.\,\lambda t'.\,\lambda m'.\,\kappa\,v'\,(t''\,@\,\kappa' :: t')\,m' \qquad \text{(shallow)}$$

Abstract machine via functional derivation

We can derive it!

via defunctionalization (and stack introduction and others).

e	$\Rightarrow$	$\langle e, [], [], \mathsf{C}_0, [], (), [] \rangle_{\mathsf{e}}$
$\langle x, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle c, \mathsf{nth}\ vs\ (\mathsf{offset}\ x\ xs), s, t, m \rangle_{\mathsf{c}}$
$\langle n, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle c, n, s, t, m \rangle_{\mathsf{c}}$
$\langle \lambda x. e, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle c, \mathsf{VFun}(x, e, xs, vs), s, t, m \rangle_{\mathsf{c}}$
$\langle e_1\, e_2, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle e_1, xs, vs, \mathsf{CApp}_1(e_2, xs, c), \mathsf{VEnv}(vs) :: s, t, m \rangle_{\mathsf{e}}$
$\langle \mathsf{try}\ e_1\ \mathsf{with}\ x, k \rightarrow e_2, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle e_1, xs, vs, \mathsf{C}_0, [], (), (c, \mathsf{H}(x, k, e_2, xs, vs), s, t) :: m \rangle_{\mathsf{e}}$
$\langle \mathsf{pfm}^{\mathsf{D}} e, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle e, xs, vs, \mathsf{CPfm}^{\mathsf{D}}(c), s, t, m \rangle_{\mathsf{e}}$
$\langle \mathsf{pfm}^{\mathsf{S}} e, xs, vs, c, s, t, m \rangle_{\mathsf{e}}$	$\Rightarrow$	$\langle e, xs, vs, \mathsf{CPfm}^{\mathsf{S}}(c), s, t, m \rangle_{\mathsf{e}}$
$\langle \mathsf{C}_0, v, [], (), [] \rangle_{\mathsf{c}}$	$\Rightarrow$	v
$\langle \mathsf{C}_0, v, [], (), (c, h, s, t) :: m \rangle_{\mathsf{c}}$	$\Rightarrow$	$\langle c, v, s, t, m \rangle_{\mathsf{c}}$
$\langle \mathsf{C}_0, v, [], t, m \rangle_{\mathsf{c}}$	$\Rightarrow$	$\langle t, v, (), m \rangle_{\mathsf{t}}$
$\langle \mathsf{CApp}_1(e_2, xs, c), v_1, \mathsf{VEnv}(vs) :: s, t, m \rangle_{\mathsf{c}}$	$\Rightarrow$	$\langle e_2, xs, vs, \mathsf{CApp}_2(c), v_1 :: s, t, m \rangle_{\mathsf{e}}$
$\langle \mathsf{CApp}_2(c), v_2, v_1 :: s, t, m \rangle_{\mathsf{c}}$	$\Rightarrow$	$\langle v_1, v_2, c, s, t, m \rangle_{\mathsf{a}}$
$\langle \mathsf{CPfm}^{\mathsf{D}}(c), v, s, t, (c_0, h_0, s_0, t_0) :: m_0 \rangle_{\mathsf{c}}$	$\Rightarrow$	$\langle h_0, v, \mathsf{VCnt}^{\mathsf{D}}(c, s, t, h_0), c_0, s_0, t_0, m_0 \rangle_{\mathsf{h}}$
$\langle \mathsf{CPfm}^{\mathsf{S}}(c), v, s, t, (c_0, h_0, s_0, t_0) :: m_0 \rangle_{\mathsf{c}}$	$\Rightarrow$	$\langle h_0, v, \mathsf{VCnt}^{\mathsf{S}}(c, s, t), c_0, s_0, t_0, m_0 \rangle_{\mathsf{h}}$
$\langle \mathsf{H}(x, k, e_2, xs, vs), v, v_k, c_0, s_0, t_0, m_0 \rangle_{\mathsf{h}}$	$\Rightarrow$	$\langle e_2, x :: k :: xs, v :: v_k :: vs, c_0, s_0, t_0, m_0 \rangle_{\mathsf{e}}$
$\langle \mathsf{Hold}(c, s), v, t, m \rangle_{\mathsf{t}}$	$\Rightarrow$	$\langle c, v, s, t, m \rangle_{\mathsf{c}}$
$\langle \mathsf{Apnd}(k, k'), v, t, m \rangle_{\mathsf{t}}$	$\Rightarrow$	$\langle k, v, k' ::_{\mathsf{t}} t, m \rangle_{\mathsf{t}}$
$\langle \mathsf{VFun}(x, e, xs, vs), v, c, s, t, m \rangle_{\mathsf{a}}$	$\Rightarrow$	$\langle e, x :: xs, v :: vs, c, s, t, m \rangle_{\mathsf{e}}$
$\langle \mathsf{VCnt}^{\mathsf{D}}(c', s', t', h), v, c, s, t, m \rangle_{\mathsf{a}}$	$\Rightarrow$	$\langle c', v, s', t', (c, h, s, t) :: m \rangle_{\mathsf{c}}$
$\langle \mathsf{VCnt}^{\mathsf{S}}(c', s', t'), v, c, s, t, m \rangle_{\mathsf{a}}$	$\Rightarrow$	$\langle c', v, s', t' @_{\mathsf{t}} (\mathsf{Hold}(c, s) ::_{\mathsf{t}} t), m \rangle_{\mathsf{c}}$

Type system with answer/trail type modification

We can extract it!

by reading off types from the interpreter.

$$\Gamma \vdash e : \tau \langle \mu_\alpha, \sigma_\alpha \rangle \alpha \langle \mu_\beta, \sigma_\beta \rangle \beta$$

$$\begin{array}{rcl} \tau \rightarrow \langle \mu_\alpha, \sigma_\alpha \rangle \alpha & : & \text{type of continuations} \\ \mu_\alpha, \mu_\beta & : & \text{type of trails} \\ \sigma_\alpha, \sigma_\beta & : & \text{type of metacontinuations} \\ \alpha, \beta & : & \text{answer types} \end{array}$$

$$\mathcal{E}\, [\![e]\!]\, \rho\, \kappa^{\tau \rightarrow \langle \mu_\alpha, \sigma_\alpha \rangle \alpha} \;=\; \lambda t^{\mu_\beta}.\, \lambda m^{\sigma_\beta}.\, e^{\beta}$$

Benefit of the functional derivation approach

- We can mechanically derive various artifacts.
- Correctness comes free from correctness of program transformations.

It forces us to think how the definitional interpreter should be.

- It is often more difficult to come up with a concise, HO, definition than to create a low-level implementation.
- Functional derivation approach leads us to the essence of the language.